Computing Probability of Symbol Error

- When decision boundaries intersect at right angles, then it is possible to compute the error probability exactly in closed form.
  - The result will be in terms of the Q-function.
  - This happens whenever the signal points form a rectangular grid in signal space.
  - Examples: QPSK and 16-QAM
- When decision regions are not rectangular, then closed form expressions are not available.
  - Computation requires integrals over the Q-function.
  - We will derive good bounds on the error rate for these cases.
  - For exact results, numerical integration is required.

Illustration: 2-dimensional Rectangle

- Assume that the n-th signal was transmitted and that the representation for this signal is $\vec{s}_n = (s_{n,0}, s_{n,1})'$.
- Assume that the decision region Γ_n is a rectangle

$$\Gamma_n = \{\vec{r} = (r_0, r_1)' : s_{n,0} - a_1 < r_0 < s_{n,0} + a_2 \text{ and } s_{n,1} - b_1 < r_1 < s_{n,1} + b_2\}.$$

  - Note: we have assumed that the sides of the rectangle are parallel to the axes in signal space.
  - Since *rotation* and *translation* of signal space do not affect distances this can be done without affecting the error probability.
- **Question:** What is the conditional error probability, assuming that $s_n(t)$ was sent.

Illustration: 2-dimensional Rectangle

- In terms of the random variables $R_k = \langle R_t, \Phi_k \rangle$, with $k = 0, 1$, an error occurs if

$$\overbrace{(R_0 \leq s_{n,0} - a_1 \text{ or } R_0 \geq s_{n,0} + a_2)}^{\text{error event 1}} \text{ or}$$
$$\underbrace{(R_1 \leq s_{n,1} - b_1 \text{ or } R_1 \geq s_{n,1} + b_2)}_{\text{error event 2}}.$$

- Note that the two error events are not mutually exclusive.
- Therefore, it is better to consider correct decisions instead, i.e., $\vec{R} \in \Gamma_n$:

$$s_{n,0} - a_1 < R_0 < s_{n,0} + a_2 \textbf{ and } s_{n,1} - b_1 < R_1 < s_{n,1} + b_2$$

Illustration: 2-dimensional Rectangle

- We know that R_0 and R_1 are
  - independent - because Φ_k are orthogonal
  - with means $s_{n,0}$ and $s_{n,1}$, respectively
  - variance $\frac{N_0}{2}$.
- Hence, the probability of a correct decision is

$$
\begin{aligned}
\Pr\{c|s_n\} &= \Pr\{-a_1 < N_0 < a_2\} \cdot \Pr\{-b_1 < N_1 < b_2\} \\
&= \int_{-a_1}^{a_2} p_{R_0|s_n}(r_0)\, dr_0 \cdot \int_{-b_1}^{b_2} p_{R_1|s_n}(r_1)\, dr_1 \\
&= (1 - Q\left(\frac{a_1}{\sqrt{N_0/2}}\right) - Q\left(\frac{a_2}{\sqrt{N_0/2}}\right)) \cdot \\
&\quad (1 - Q\left(\frac{b_1}{\sqrt{N_0/2}}\right) - Q\left(\frac{b_2}{\sqrt{N_0/2}}\right)).
\end{aligned}
$$

Exercise: QPSK

- Find the error rate for the signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n\cdot\pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3.$$

- **Answer:** (Recall $\eta_P = \frac{d^2_{\min}}{E_b} = 4$ for QPSK)

$$\begin{aligned}
\Pr\{e\} &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \\
&= 2Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$

Exercise: 16-QAM

(Recall $\eta_P = \frac{d^2_{\min}}{E_b} = \frac{5}{3}$ for 16-QAM)

- Find the error rate for the signal set ($a_I, a_Q \in \{-3, -1, 1, 3\}$)

$$s_n(t) = \sqrt{2E_0/T} a_I \cdot \cos(2\pi f_c t) + \sqrt{2E_0/T} a_Q \cdot \sin(2\pi f_c t)$$

- **Answer:** ($\eta_P = \frac{d^2_{\min}}{E_b} = 4$)

$$\begin{aligned}
\Pr\{e\} &= 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{2E_0}{N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right) \\
&= 3Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$

N-dimensional Hypercube

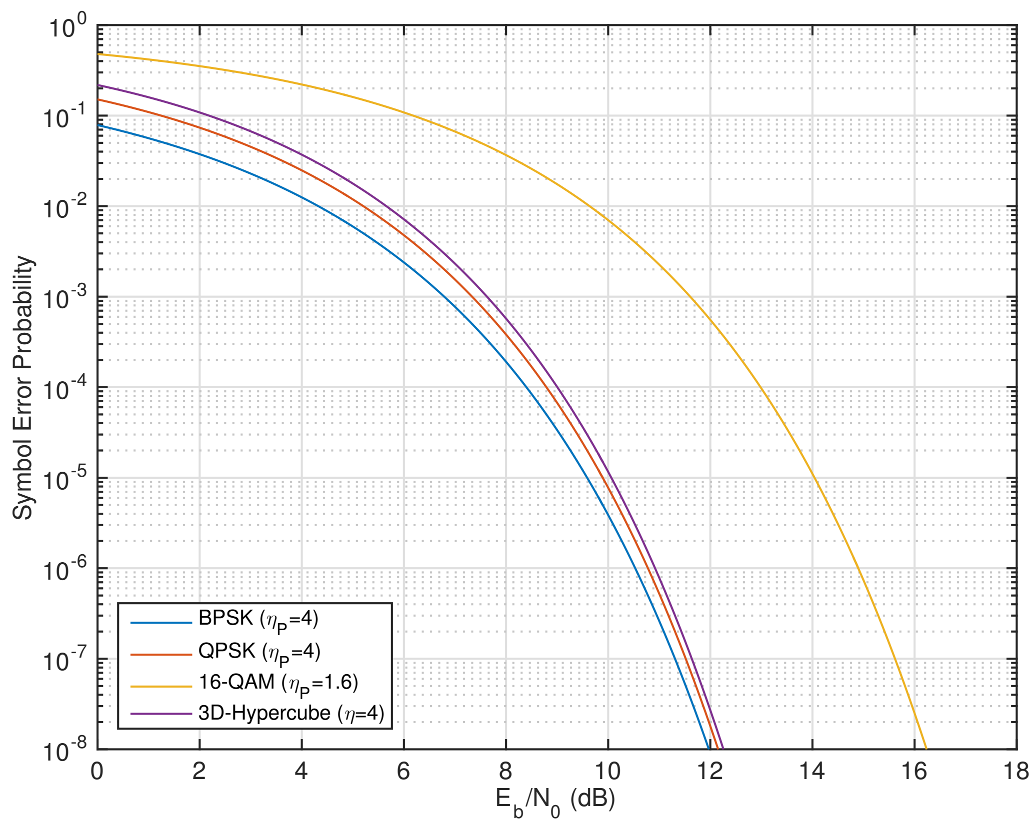
- Find the error rate for the signal set with 2^N signals of the form ($b_{k,n} \in \{-1, 1\}$):

$$s_n(t) = \sum_{k=1}^{N} \sqrt{\frac{2E_s}{NT}} b_{k,n} \cos(2\pi nt/T), \text{ for } 0 \leq t \leq T$$

- **Answer:**

$$\begin{aligned}
\Pr\{e\} &= 1 - \left(1 - Q\left(\sqrt{\frac{2E_s}{N \cdot N_0}}\right)\right)^N \\
&= 1 - \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^N \\
&= 1 - \left(1 - Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right)\right)^N \approx N \cdot Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).
\end{aligned}$$

Comparison



- Better power efficiency η_P leads to better error performance (at high SNR).

What if Decision Regions are not Rectangular?

- **Example:** For 8-PSK, the probability of a correct decision is given by the following integral over the decision region for $s_0(t)$

$$\Pr\{c\} = \int_0^\infty \frac{1}{\sqrt{2\pi N_0/2}} \exp(-\frac{(x-\sqrt{E_s})^2}{2N_o/2} \underbrace{\int_{-x\tan(\pi/8)}^{x\tan(\pi/8)} \frac{1}{\sqrt{2\pi N_0/2}} \exp(-\frac{y^2}{2N_0/2})\, dy}_{=1-2Q(\frac{x\tan(\pi/8)}{\sqrt{N_0/2}})}\, dx$$

- This integral cannot be computed in closed form.

Union Bound

- When decision boundaries do not intersect at right angles, then the error probability cannot be computed in closed form.
- An upper bound on the conditional probability of error (assuming that s_n was sent) is provided by:

$$\Pr\{e|s_n\} \leq \sum_{k \neq n} \Pr\{\|\vec{R} - \vec{s}_k\| < \|\vec{R} - \vec{s}_n\| | \vec{s}_n\}$$
$$= \sum_{k \neq n} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{2\sqrt{N_0/2}}\right).$$

  - Note that this bound is computed from *pairwise error probabilities* between s_n and all other signals.

Union Bound

- Then, the average probability of error can be bounded by

$$\Pr\{e\} = \sum_n \pi_n \sum_{k \neq n} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{\sqrt{2N_0}}\right).$$

- This bound is called the union bound; it approximates the union of all possible error events by the sum of the pairwise error probabilities.

Example: QPSK

- For the QPSK signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n \cdot \pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3$$

the union bound is

$$\Pr\{e\} \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

  - Recall that the exact probability of error is

$$\Pr\{e\} = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right).$$

"Intelligent" Union Bound

- The union bound is easily tightened by recognizing that only immediate neighbors of s_n must be included in the bound on the conditional error probability.
- Define the the neighbor set $N_{ML}(s_n)$ of s_n as the set of signals s_k that share a decision boundary with signal s_n.
- Then, the conditional error probability is bounded by

$$\begin{aligned}\Pr\{e|s_n\} &\leq \sum_{k \in N_{ML}(s_n)} \Pr\{\|\vec{R} - \vec{s}_k\| < \|\vec{R} - \vec{s}_n\| | \vec{s}_n\} \\ &= \sum_{k \in N_{ML}(s_n)} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{2\sqrt{N_0/2}}\right).\end{aligned}$$

"Intelligent" Union Bound

- Then, the average probability of error can be bounded by

$$\Pr\{e\} \leq \sum_n \pi_n \sum_{k \in N_{ML}(s_n)} Q\left(\frac{\|\vec{s}_k - \vec{s}_n\|}{\sqrt{2N_0}}\right).$$

- We refer to this bound as the intelligent union bound.
  - It still relies on pairwise error probabilities.
  - It excludes many terms in the union bound; thus, it is tighter.

Example: QPSK

- For the QPSK signal set

$$s_n(t) = \sqrt{2E_s/T}\cos(2\pi f_c t + n\cdot\pi/2 + \pi/4), \text{ for } n = 0, \ldots, 3$$

the intelligent union bound includes only the immediate neighbors of each signal:

$$\Pr\{e\} \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right).$$

  - Recall that the exact probability of error is

$$\Pr\{e\} = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right).$$

Example: 16-QAM

- For the 16-QAM signal set, there are
  - 4 signals s_i that share a decision boundary with 4 neighbors; bound on conditional error probability: $\Pr\{e|s_i\} = 4Q(\sqrt{\frac{2E_0}{N_0}})$.
  - 8 signals s_c that share a decision boundary with 3 neighbors; bound on conditional error probability: $\Pr\{e|s_c\} = 3Q(\sqrt{\frac{2E_0}{N_0}})$.
  - 4 signals s_o that share a decision boundary with 2 neighbors; bound on conditional error probability: $\Pr\{e|s_o\} = 2Q(\sqrt{\frac{2E_0}{N_0}})$.
- The resulting intelligent union bound is

$$\Pr\{e\} \leq 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right).$$

Example: 16-QAM

- The resulting intelligent union bound is

$$\Pr\{e\} \leq 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right).$$

  - Recall that the exact probability of error is

$$\Pr\{e\} = 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{2E_0}{N_0}}\right).$$

Nearest Neighbor Approximation

- At high SNR, the error probability is dominated by terms that involve the shortest distance d_{min} between any pair of nodes.
  - The corresponding error probability is proportional to $Q(\sqrt{\frac{d_{\text{min}}}{2N_0}})$.
- For each signal s_n, we count the number N_n of neighbors at distance d_{min}.
- Then, the error probability at high SNR can be approximated as

$$\Pr\{e\} \approx \frac{1}{M} \sum_{n=1}^{M-1} N_n Q(\sqrt{\frac{d_{\text{min}}}{2N_0}}) = \bar{N}_{\text{min}} Q(\sqrt{\frac{d_{\text{min}}}{2N_0}}).$$

Example: 16-QAM

- In 16-QAM, the distance between adjacent signals is $d_{\text{min}} = 2\sqrt{E_s}$.
- There are:
  - 4 signals with 4 nearest neighbors
  - 8 signals with 3 nearest neighbors
  - 4 signals with 2 nearest neighbors
- The average number of neighbors is $\bar{N}_{\text{min}} = 3$.
- The error probability is approximately,

$$\Pr\{e\} \approx 3Q\left(\sqrt{\frac{2E_0}{N_0}}\right).$$

  - same as the intelligent union bound.

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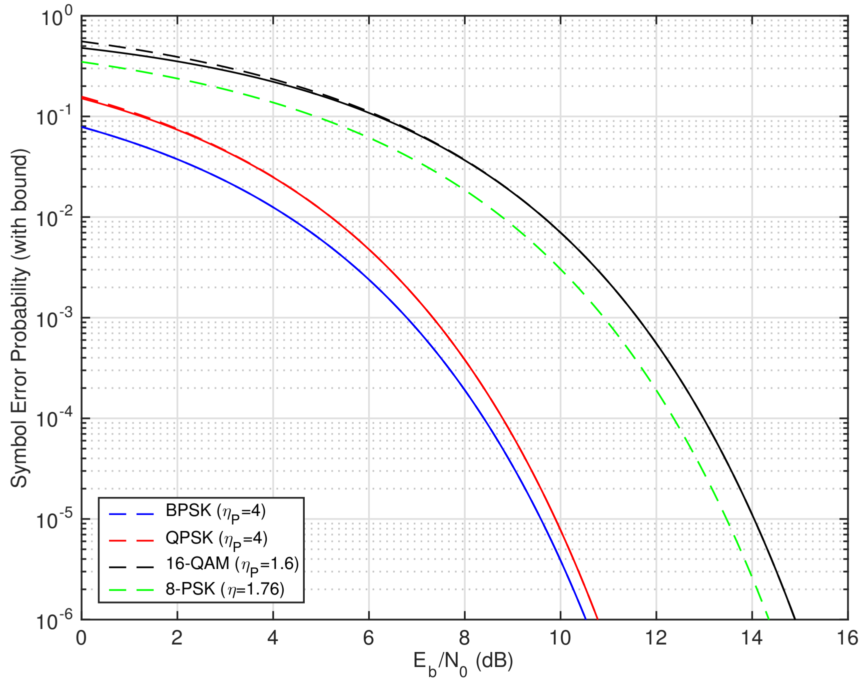
Example: 8-PSK

- For 8-PSK, each signal has 2 nearest neighbors at distance $d_{\min} = \sqrt{(2-\sqrt{2})E_s}$.
- Hence, both the intelligent union bound and the nearest neighbor approximation yield

$$\Pr\{e\} \approx 2Q\left(\sqrt{\frac{(2-\sqrt{2})E_s}{2N_0}}\right) = 2Q\left(\sqrt{\frac{3(2-\sqrt{2})E_b}{2N_0}}\right)$$

  - Since, $E_b = 3E_s$.

Comparison



Solid: exact P_e, dashed: approximation. For 8PSK, only approximation is shown.

- The intelligent union bound is very tight for all cases considered here.
  - It also coincides with the nearest neighbor approximation

General Approximation for Probability of Symbol Error

- From the above examples, we can conclude that a good, general approximation for the probability of error is given by

$$\Pr\{e\} \approx \bar{N}_{\min} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = \bar{N}_{\min} Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right).$$

- Probability of error depends on
  - signal-to-noise ratio (SNR) E_b/N_0 and
  - geometry of the signal constellation via the average number of neighbors $\bar{N}_{\min}$ and the power efficiency η_P.

Bit Errors

- So far, we have focused on symbol errors; however, ultimately we are concerned about bit errors.
- There are many ways to map groups of $\log_2(M)$ bits to the M signals in a constellation.
- **Example QPSK:** Which mapping is better?

QPSK Phase	Mapping 1	Mapping 2
$\pi/4$	00	00
$3\pi/4$	01	01
$5\pi/4$	10	11
$7\pi/4$	11	10

Bit Errors

- **Example QPSK:**

QPSK Phase	Mapping 1	Mapping 2
$\pi/4$	00	00
$3\pi/4$	01	01
$5\pi/4$	10	11
$7\pi/4$	11	10

- Note, that for Mapping 2 *nearest neighbors* differ in exactly one bit position.
  - That implies, that the most common symbol errors will induce only one bit error.
  - That is not true for Mapping 1.

Gray Coding

- A mapping of $log_2(M)$ bits to M signals is called Gray Coding if
  - The bit patterns that are assigned to nearest neighbors in the constelation
  - differ in exactly one bit position.
- With Gray coding, the most likely symbol errors induce exactly one bit error.
  - Note that there are $\log_2(M)$ bits for each symbol.
- Hence, with Gray coding the *bit error probability* is well approximated by

$$\Pr\{\text{bit error}\} \approx \frac{\bar{N}_{\min}}{\log_2(M)} Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) \lesssim Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right).$$