Part II

Mathematical Prerequisites

Gaussian Random Variables — Why we Care

- Gaussian random variables play a critical role in modeling many random phenomena.
  - By central limit theorem, Gaussian random variables arise from the superposition (sum) of many random phenomena.
    - Pertinent example: random movement of very many electrons in conducting material.
    - Result: thermal noise is well modeled as Gaussian.
  - Gaussian random variables are mathematically tractable.
    - In particular: any linear (more precisely, affine) transformation of Gaussians produces a Gaussian random variable.
- Noise added by channel is modeled as being Gaussian.
  - Channel noise is the most fundamental impairment in a communication system.

Gaussian Random Variables

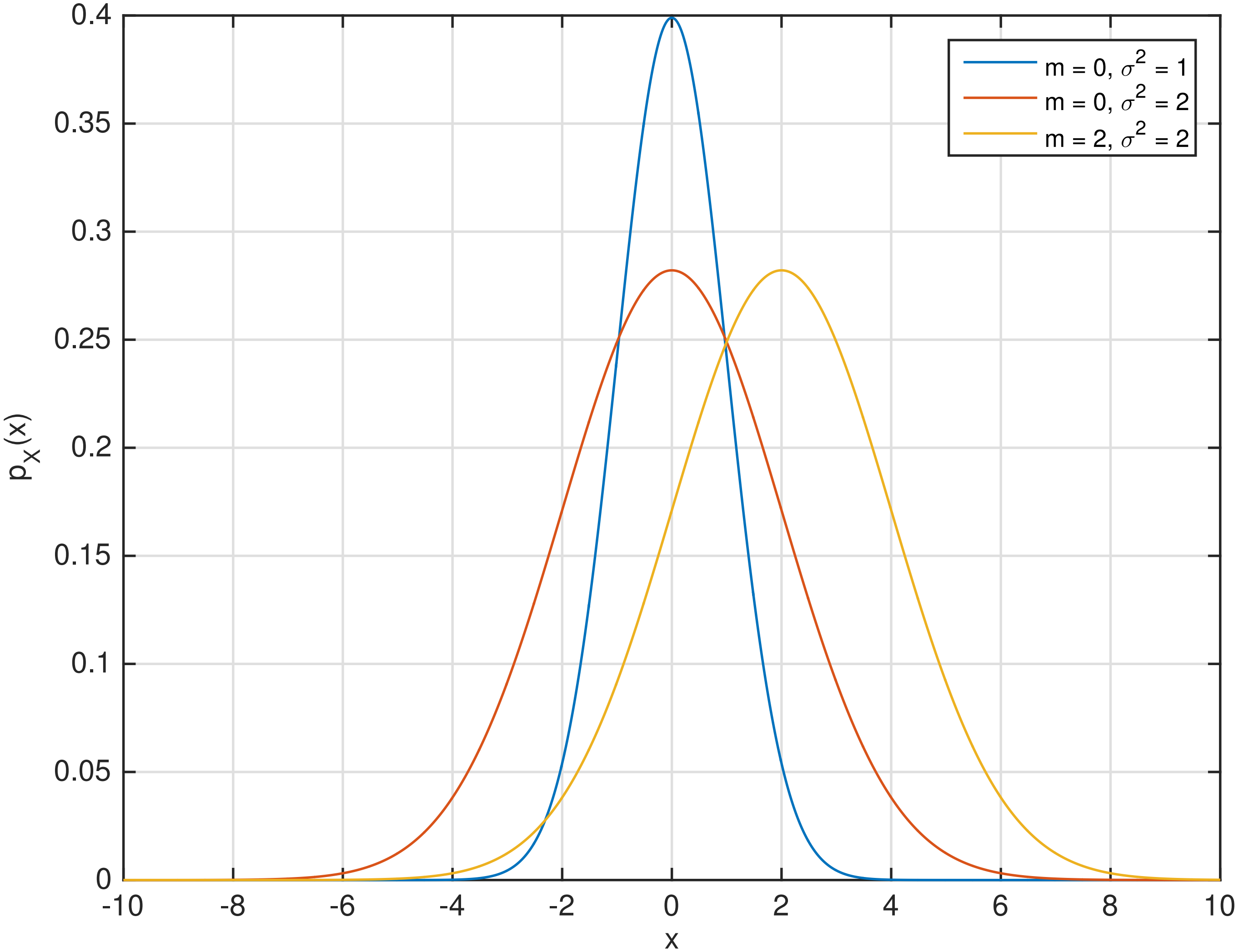
- A random variable X is said to be Gaussian (or Normal) if its pdf is of the form

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

- All properties of a Gaussian are determined by the two parameters m and σ^2.
- **Notation:** $X \sim \mathcal{N}(m, \sigma^2)$.
- **Moments:**

$$\begin{aligned} \mathbf{E}[X] &= \int_{-\infty}^{\infty} x \cdot p_X(x)\,dx = m \\ \mathbf{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot p_X(x)\,dx = m^2 + \sigma^2. \end{aligned}$$

Plot of Gaussian pdf's



The Gaussian Error Integral — Q(x)

- We are often interested in $\Pr\{X > x\}$ for Gaussian random variables X.
- These probabilities cannot be computed in closed form since the integral over the Gaussian pdf does not have a closed form expression.
- Instead, these probabilities are expressed in terms of the Gaussian error integral

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

The Gaussian Error Integral — Q(x)

- **Example:** Suppose $X \sim \mathcal{N}(1,4)$, what is $\Pr\{X > 5\}$?

$$\begin{aligned}\Pr\{X > 5\} &= \int_5^\infty \frac{1}{\sqrt{2\pi \cdot 2^2}} e^{-\frac{(x-1)^2}{2\cdot 2^2}}\, dx \qquad \text{substitute } z = \frac{x-1}{2} \\ &= \int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = Q(2)\end{aligned}$$

Exercises

- Let $X \sim \mathcal{N}(-3, 4)$, find expressions in terms of $Q(\cdot)$ for the following probabilities:
  1. $\Pr\{X > 5\}$?
  2. $\Pr\{X < -1\}$?
  3. $\Pr\{X^2 + X > 2\}$?

Bounds for the Q-function

- Since no closed form expression is available for $Q(x)$, bounds and approximations to the Q-function are of interest.
- The following bounds are tight for large values of x:

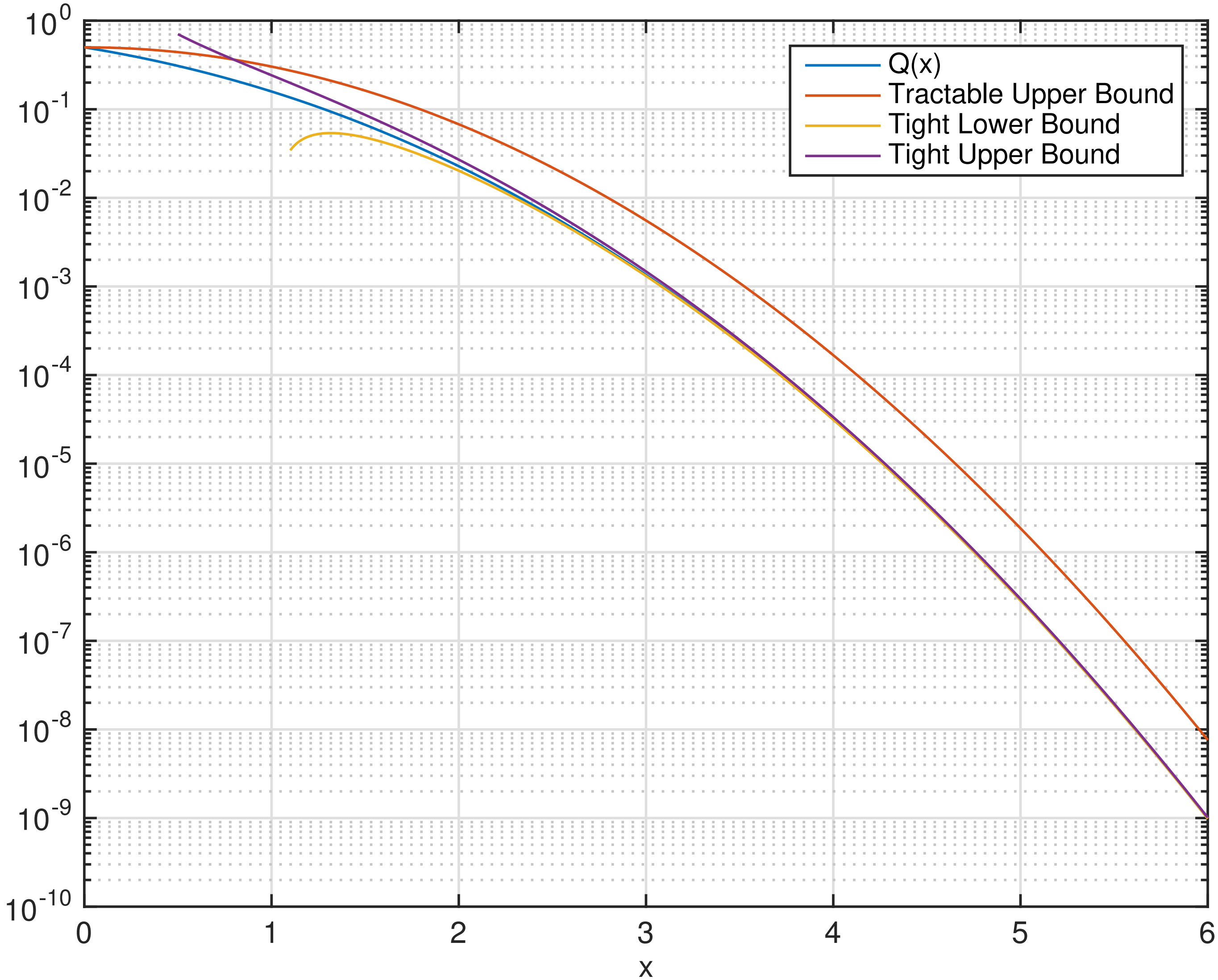
$$\left(1-\frac{1}{x^2}\right)\frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}.$$

- The following bound is not as quite as tight but very useful for analysis

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}.$$

- Note that all three bounds are dominated by the term $e^{-\frac{x^2}{2}}$; this term determines the asymptotic behaviour of $Q(x)$.

Plot of Q(x) and Bounds



Gaussian Random Vectors

- A length N random vector $\vec{X}$ is said to be Gaussian if its pdf is given by

$$p_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2}|K|^{1/2}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{m})^T K^{-1}(\vec{x}-\vec{m})\right).$$

- **Notation:** $\vec{X} \sim \mathcal{N}(\vec{m}, K)$.
- Mean vector

$$\vec{m} = \mathbf{E}[\vec{X}] = \int_{-\infty}^{\infty} \vec{x} p_{\vec{X}}(\vec{x})\, d\vec{x}.$$

- Covariance matrix

$$K = \mathbf{E}[(\vec{X}-\vec{m})(\vec{X}-\vec{m})^T] = \int_{-\infty}^{\infty} (\vec{x}-\vec{m})(\vec{x}-\vec{m})^T p_{\vec{X}}(\vec{x})\, d\vec{x}.$$

  - $|K|$ denotes the determinant of K.
  - K must be positive definite, i.e., $\vec{z}^T K \vec{z} > 0$ for all $\vec{z}$.

Exercise: Important Special Case: N=2

- Consider a length-2 Gaussian random vector with

$$\vec{m} = \vec{0} \text{ and } K = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

- Find the pdf of $\vec{X}$.
- Answer:

$$p_{\vec{X}}(\vec{x}) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2\sigma^2(1-\rho^2)}\right)$$

Important Properties of Gaussian Random Vectors

1. If the N Gaussian random variables X_n comprising the random vector $\vec{X}$ are uncorrelated ($\text{Cov}[X_i, X_j] = 0$, for $i \neq j$), then they are statistically independent.
2. Any affine transformation of a Gaussian random vector is also a Gaussian random vector.
   - Let $\vec{X} \sim \mathcal{N}(\vec{m}, K)$
   - Affine transformation: $\vec{Y} = A\vec{X} + \vec{b}$
   - Then, $\vec{Y} \sim \mathcal{N}(A\vec{m} + \vec{b}, AKA^T)$

Exercise: Generating Correlated Gaussian Random Variables

- Let $\vec{X} \sim \mathcal{N}(\vec{m}, K)$, with

$$\vec{m} = \vec{0} \text{ and } K = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  - The elements of $\vec{X}$ are uncorrelated.
- Transform $\vec{Y} = A\vec{X}$, with

$$A = \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix}$$

- Find the pdf of $\vec{Y}$.

Random Processes — Why we Care

- Random processes describe signals that change randomly over time.
  - Compare: deterministic signals can be described by a mathematical expression that describes the signal exactly for all time.
  - Example: $x(t) = 3\cos(2\pi f_c t + \pi/4)$ with $f_c = 1$GHz.
- We will encounter three types of random processes in communication systems:
  1. (nearly) deterministic signal with a random parameter — Example: sinusoid with random phase.
  2. signals constructed from a sequence of random variables — Example: digitally modulated signals with random symbols
  3. noise-like signals
- **Objective:** Develop a framework to describe and analyze random signals encountered in the receiver of a communication system.

Random Process — Formal Definition

- Random processes can be defined completely analogous to random variables over a probability triple space $(\Omega, \mathcal{F}, P)$.
- **Definition:** A random process is a mapping from each element ω of the sample space Ω to a function of time (i.e., a signal).
- Notation: $X_t(\omega)$ — we will frequently omit ω to simplify notation.
- Observations:
  - We will be interested in both real and complex valued random processes.
  - Note, for a given random outcome ω_0, $X_t(\omega_0)$ is a *deterministic* signal.
  - Note, for a fixed time t_0, $X_{t_0}(\omega)$ is a *random variable*.

Sample Functions and Ensemble

- For a given random outcome ω_0, $X_t(\omega_0)$ is a deterministic signal.
  - Each signal that that can be produced by a our random process is called a sample function of the random process.
- The collection of all sample functions of a random process is called the ensemble of the process.
- **Example:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
  The ensemble of this random process consists of the four sample functions:

$$X_t(\omega_1) = \cos(2\pi f_0 t) \qquad X_t(\omega_2) = -\sin(2\pi f_0 t)$$
$$X_t(\omega_3) = -\cos(2\pi f_0 t) \qquad X_t(\omega_4) = \sin(2\pi f_0 t)$$

Probability Distribution of a Random Process

- For a given time instant t, $X_t(\omega)$ is a random variable.
- Since it is a random variable, it has a pdf (or pmf in the discrete case).
  - We denote this pdf as $p_{X_t}(x)$.
- The statistical properties of a random process are specified completely if the joint pdf

$$p_{X_{t_1},\ldots,X_{t_n}}(x_1,\ldots,x_n)$$

is available for all n and t_i, $i = 1,\ldots,n$.
  - This much information is often not available.
  - Joint pdfs with many sampling instances can be cumbersome.
  - We will shortly see a more concise summary of the statistics for a random process.

Random Process with Random Parameters

- A deterministic signal that depends on a random parameter is a random process.
  - Note, the sample functions of such random processes do not "look" random.
- Running Examples:
  - **Example (discrete phase):** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
  - **Example (continuous phase):** same as above but phase $\Theta(\omega)$ is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim U[0, 2\pi)$.
- For both of these processes, the complete statistical description of the random process can be found.

Example: Discrete Phase Process

- **Discrete Phase Process:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- Find the first-order density $p_{X_t}(x)$ for this process.
- Find the second-order density $p_{X_{t_1} X_{t_2}}(x_1, x_2)$ for this process.
  - Note, since the phase values are discrete the above pdfs must be expressed with the help of δ-functions.
  - Alternatively, one can derive a probability mass function.

Solution: Discrete Phase Process

- First-order density function:

$$p_{X_t}(x) = \frac{1}{4}(\delta(x - \cos(2\pi f_0 t)) + \delta(x + \sin(2\pi f_0 t)) + \delta(x + \cos(2\pi f_0 t)) + \delta(x - \sin(2\pi f_0 t)))$$

- Second-order density function:

$$\begin{aligned} p_{X_{t_1} X_{t_2}}(x_1, x_2) = \frac{1}{4}(&\delta(x_1 - \cos(2\pi f_0 t_1)) \cdot \delta(x_2 - \cos(2\pi f_0 t_2)) + \\ &\delta(x_1 + \sin(2\pi f_0 t_1)) \cdot \delta(x_2 + \sin(2\pi f_0 t_2)) + \\ &\delta(x_1 + \cos(2\pi f_0 t_1)) \cdot \delta(x_2 + \cos(2\pi f_0 t_2)) + \\ &\delta(x_1 - \sin(2\pi f_0 t_1)) \cdot \delta(x_2 - \sin(2\pi f_0 t_2))) \end{aligned}$$

Example: Continuous Phase Process

- **Continuous Phase Process:** Let $\Theta(\omega)$ be a random variable that is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim [0, 2\pi)$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- Find the first-order density $p_{X_t}(x)$ for this process.
- Find the second-order density $p_{X_{t_1} X_{t_2}}(x_1, x_2)$ for this process.

Solution: Continuous Phase Process

- First-order density:

$$p_{X_t}(x) = \frac{1}{\pi\sqrt{1-x^2}} \qquad \text{for } |x| \leq 1.$$

  Notice that $p_{X_t}(x)$ does **not** depend on t.

- Second-order density:

$$p_{X_{t_1}X_{t_2}}(x_1, x_2) = \frac{1}{\pi\sqrt{1-x_2^2}} \cdot [\frac{1}{2} \cdot$$
$$\delta(x_1 - \cos(2\pi f_0(t_1 - t_2) + \arccos(x_2))) +$$
$$\delta(x_1 - \cos(2\pi f_0(t_1 - t_2) - \arccos(x_2)))]$$

Random Processes Constructed from Sequence of Random Experiments

- Model for digitally modulated signals.
- Example:
  - Let $X_k(\omega)$ denote the outcome of the k-th toss of a coin:

$$X_k(\omega) = \begin{cases} 1 & \text{if heads on } k\text{-th toss} \\ -1 & \text{if tails on } k\text{-th toss.} \end{cases}$$

  - Let $p(t)$ denote a pulse of duration T, e.g.,

$$p(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T \\ 0 & \text{else.} \end{cases}$$

  - Define the random process X_t

$$X_t(\omega) = \sum_k X_k(\omega) p(t - nT)$$

Probability Distribution

- Assume that heads and tails are equally likely.
- Then the first-order density for the above random process is

$$p_{X_t}(x) = \frac{1}{2}(\delta(x-1) + \delta(x+1)).$$

- The second-order density is:

$$p_{X_{t_1}X_{t_2}}(x_1, x_2) = \begin{cases} \delta(x_1 - x_2) p_{X_{t_1}}(x_1) & \text{if } nT \leq t_1, t_2 \leq (n+1)T \\ p_{X_{t_1}}(x_1) p_{X_{t_2}}(x_2) & \text{else.} \end{cases}$$

  - These expression become more complicated when $p(t)$ is not a rectangular pulse.

Probability Density of Random Processs Defined Directly

- Sometimes the n-th order probability distribution of the random process is given.
  - Most important example: Gaussian Random Process
    - Statistical model for noise.
  - **Definition:** The random process X_t is Gaussian if the vector $\vec{X}$ of samples taken at times $t_1, \ldots, t_n$

$$\vec{X} = \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}$$

is a Gaussian random vector for all $t_1, \ldots, t_n$.

Second Order Description of Random Processes

- Characterization of random processes in terms of n-th order densities is
  - frequently not available
  - mathematically cumbersome
- A more tractable, practical alternative description is provided by the second order description for a random process.
- **Definition:** The second order description of a random process consists of the
  - mean function and the
  - autocorrelation function

  of the process.
- Note, the second order description can be computed from the (second-order) joint density.
  - The converse is not true — at a minimum the distribution must be specified (e.g., Gaussian).

Mean Function

- The second order description of a process relies on the mean and autocorrelation functions — these are defined as follows
- **Definition:** The mean of a random process is defined as:

$$\mathbf{E}[X_t] = m_X(t) = \int_{-\infty}^{\infty} x \cdot p_{X_t}(x)\, dx$$

  - Note, that the mean of a random process is a deterministic signal.
  - The mean is computed from the first oder density function.

Autocorrelation Function

- **Definition:** The autocorrelation function of a random process is defined as:

$$R_X(t,u) = \mathbf{E}[X_t X_u] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p_{X_t,X_u}(x,y)\,dx\,dy$$

  - Autocorrelation is computed from second order density

Autocovariance Function

- Closely related: autocovariance function:

$$
\begin{aligned}
C_X(t,u) &= \mathbf{E}[(X_t - m_X(t))(X_u - m_X(u))] \\
&= R_X(t,u) - m_X(t)m_X(u)
\end{aligned}
$$

Exercise: Discrete Phase Example

- Find the second-order description for the discrete phase random process.
  - **Discrete Phase Process:** Let $\Theta(\omega)$ be a random variable with four equally likely, possible values $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- **Answer:**
  - Mean: $m_X(t) = 0$.
  - Autocorrelation function:

$$R_X(t,u) = \frac{1}{2}\cos(2\pi f_0(t-u)).$$

Exercise: Continuous Phase Example

- Find the second-order description for the continuous phase random process.
  - **Continuous Phase Process:** Let $\Theta(\omega)$ be a random variable that is uniformly distributed between 0 and 2π, $\Theta(\omega) \sim [0, 2\pi)$. Define the random process $X_t(\omega) = \cos(2\pi f_0 t + \Theta(\omega))$.
- **Answer:**
  - Mean: $m_X(t) = 0$.
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$$R_X(t,u) = \frac{1}{2}\cos(2\pi f_0(t-u)).$$