Properties of the Autocorrelation Function

- The autocorrelation function of a (real-valued) random process satisfies the following properties:
  1. $R_X(t, t) \geq 0$
  2. $R_X(t, u) = R_X(u, t)$ (symmetry)
  3. $|R_X(t, u)| \leq \frac{1}{2}(R_X(t, t) + R_X(u, u))$
  4. $|R_X(t, u)|^2 \leq R_X(t, t) \cdot R_X(u, u)$

Stationarity

- The concept of stationarity is analogous to the idea of time-invariance in linear systems.
- **Interpretation:** For a stationary random process, the statistical properties of the process do not change with time.
- **Definition:** A random process X_t is strict-sense stationary (sss) to the n-th order if:

$$p_{X_{t_1},\ldots,X_{t_n}}(x_1,\ldots,x_n) = p_{X_{t_1+T},\ldots,X_{t_n+T}}(x_1,\ldots,x_n)$$

  for all T.
  - The statistics of X_t do not depend on *absolute* time but only on the time differences between the sample times.

Wide-Sense Stationarity

- A simpler and more tractable notion of stationarity is based on the second-order description of a process.
- **Definition:** A random process X_t is wide-sense stationary (wss) if
  1. the mean function $m_X(t)$ is constant **and**
  2. the autocorrelation function $R_X(t, u)$ depends on t and u only through $t - u$, i.e., $R_X(t, u) = R_X(t - u)$
- **Notation:** for a wss random process, we write the autocorrelation function in terms of the single time-parameter $\tau = t - u$:

$$R_X(t, u) = R_X(t - u) = R_X(\tau).$$

Exercise: Stationarity

- **True or False:** Every random process that is strict-sense stationarity to the second order is also wide-sense stationary.
  - **Answer:** True
- **True or False:** Every random process that is wide-sense stationary must be strict-sense stationarity to the second order.
  - **Answer:** False
- **True or False:** The discrete phase process is strict-sense stationary.
  - **Answer:** False; first order density depends on t, therefore, not even first-order sss.
- **True or False:** The discrete phase process is wide-sense stationary.
  - **Answer:** True

White Gaussian Noise

- **Definition:** A (real-valued) random process X_t is called white Gaussian Noise if
  - X_t is Gaussian for each time instance t
  - Mean: $m_X(t) = 0$ for all t
  - Autocorrelation function: $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$
  - White Gaussian noise is a good model for noise in communication systems.
  - Note, that the variance of X_t is infinite:

$$\mathrm{Var}(X_t) = \mathbf{E}[X_t^2] = R_X(0) = \frac{N_0}{2}\delta(0) = \infty.$$

  - Also, for $t \neq u$: $\mathbf{E}[X_t X_u] = R_X(t, u) = R_X(t - u) = 0$.

Integrals of Random Processes

- We will see, that receivers always include a linear, time-invariant system, i.e., a filter.
- Linear, time-invariant systems *convolve* the input random process with the impulse response of the filter.
  - Convolution is fundamentally an integration.
- We will establish conditions that ensure that an expression like

$$Z(\omega) = \int_a^b X_t(\omega) h(t) \, dt$$

  is "well-behaved".
  - The result of the (definite) integral is a random variable.
- **Concern:** Does the above integral *converge*?

Mean Square Convergence

- There are different senses in which a sequence of random variables may converge: *almost surely*, *in probability*, *mean square*, and *in distribution*.
- We will focus exclusively on mean square convergence.
- For our integral, mean square convergence means that the Rieman sum and the random variable Z satisfy:
  - Given $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\mathbf{E}\left[\left(\sum_{k=1}^{n} X_{\tau_k} h(\tau_k)(t_k - t_{k-1}) - Z\right)^2\right] \leq \epsilon.$$

with:

- $a = t_0 < t_1 < \cdots < t_n = b$
- $t_{k-1} \leq \tau_k \leq t_k$
- $\delta = \max_k(t_k - t_{k-1})$

Mean Square Convergence — Why We Care

- It can be shown that the integral converges if

$$\int_a^b \int_a^b R_X(t,u) h(t) h(u) \, dt \, du < \infty$$

- **Important:** When the integral converges, then the order of integration and expectation can be interchanged, e.g.,

$$\mathbf{E}[Z] = \mathbf{E}[\int_a^b X_t h(t) \, dt] = \int_a^b \mathbf{E}[X_t] h(t) \, dt = \int_a^b m_X(t) h(t) \, dt$$

- Throughout this class, we will focus exclusively on cases where $R_X(t,u)$ and $h(t)$ are such that our integrals converge.

Exercise: Brownian Motion

- **Definition:** Let N_t be white Gaussian noise with $\frac{N_0}{2} = \sigma^2$. The random process

$$W_t = \int_0^t N_s \, ds \quad \text{for } t \geq 0$$

is called Brownian Motion or Wiener Process.

- Compute the mean and autocorrelation functions of W_t.
- **Answer:** $m_W(t) = 0$ and $R_W(t, u) = \sigma^2 \min(t, u)$

Integrals of Gaussian Random Processes

- Let X_t denote a Gaussian random process with second order description $m_X(t)$ and $R_X(t,s)$.
- Then, the integral

$$Z = \int_a^b X(t)h(t)\,dt$$

is a Gaussian random variable.
- Moreover mean and variance are given by

$$\mu = \mathbf{E}[Z] = \int_a^b m_X(t)h(t)\,dt$$

$$\begin{aligned}\mathrm{Var}[Z] &= \mathbf{E}[(Z-\mathbf{E}[Z])^2] = \mathbf{E}[(\int_a^b (X_t - m_x(t))h(t)\,dt)^2] \\ &= \int_a^b \int_a^b C_X(t,u)h(t)h(u)\,dt\,du\end{aligned}$$

Jointly Defined Random Processes

- Let X_t and Y_t be jointly defined random processes.
  - E.g., input and output of a filter.
- Then, joint densities of the form $p_{X_t Y_u}(x, y)$ can be defined.
- Additionally, second order descriptions that describe the correlation between samples of X_t and Y_t can be defined.

Crosscorrelation and Crosscovariance

- **Definition:** The crosscorrelation function $R_{XY}(t,u)$ is defined as:

$$R_{XY}(t,u) = \mathbf{E}[X_t Y_u] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy p_{X_t Y_u}(x,y)\, dx\, dy.$$

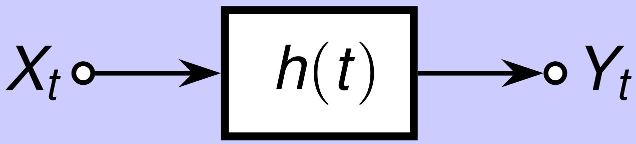
- **Definition:** The crosscovariance function $C_{XY}(t,u)$ is defined as:

$$C_{XY}(t,u) = R_{XY}(t,u) - m_X(t) m_Y(u).$$

- **Definition:** The processes X_t and Y_t are called jointly wide-sense stationary if:
  1. $R_{XY}(t,u) = R_{XY}(t-u)$ **and**
  2. $m_X(t)$ and $m_Y(t)$ are constants.

Filtering of Random Processes

Filtered Random Process



Filtering of Random Processes

- Clearly, X_t and Y_t are jointly defined random processes.
- Standard LTI system — convolution:

$$Y_t = \int h(t-\sigma) X_\sigma \, d\sigma = h(t) * X_t$$

- Recall: this convolution is "well-behaved" if

$$\int\int R_X(\sigma, \nu) h(t-\sigma) h(t-\nu) \, d\sigma \, d\nu < \infty$$

  - E.g.: $\int\int R_X(\sigma, \nu) \, d\sigma \, d\nu < \infty$ **and** $h(t)$ stable.

Second Order Description of Output: Mean

- The expected value of the filter's output Y_t is:

$$\begin{aligned}\mathbf{E}[Y_t] &= \mathbf{E}[\int h(t-\sigma)X_\sigma \, d\sigma] \\ &= \int h(t-\sigma)\mathbf{E}[X_\sigma] \, d\sigma \\ &= \int h(t-\sigma)m_X(\sigma) \, d\sigma\end{aligned}$$

- For a wss process X_t, $m_X(t)$ is constant. Therefore,

$$\mathbf{E}[Y_t] = m_Y(t) = m_X \int h(\sigma) \, d\sigma$$

is also constant.

Crosscorrelation of Input and Output

- The crosscorrelation between input and ouput signals is:

$$R_{XY}(t,u) = \mathbf{E}[X_t Y_u] = \mathbf{E}[X_t \int h(u-\sigma) X_\sigma \, d\sigma$$
$$= \int h(u-\sigma) \mathbf{E}[X_t X_\sigma] \, d\sigma$$
$$= \int h(u-\sigma) R_X(t,\sigma) \, d\sigma$$

- For a wss input process

$$R_{XY}(t,u) = \int h(u-\sigma) R_X(t,\sigma) \, d\sigma = \int h(\nu) R_X(t, u-\nu) \, d\nu$$
$$= \int h(\nu) R_X(t-u+\nu) \, d\nu = R_{XY}(t-u)$$

  - Input and output are jointly stationary.

Autocorelation of Output

- The autocorrelation of Y_t is given by

$$R_Y(t,u) = \mathbf{E}[Y_t Y_u] = \mathbf{E}[\int h(t-\sigma) X_\sigma \, d\sigma \int h(u-\nu) X_\nu \, d\nu]$$
$$= \int\int h(t-\sigma) h(u-\nu) R_X(\sigma,\nu) \, d\sigma \, d\nu$$

- For a wss input process:

$$R_Y(t,u) = \int\int h(t-\sigma) h(u-\nu) R_X(\sigma,\nu) \, d\sigma \, d\nu$$
$$= \int\int h(\lambda) h(\lambda-\gamma) R_X(t-\lambda, u-\lambda+\gamma) \, d\lambda \, d\gamma$$
$$= \int\int h(\lambda) h(\lambda-\gamma) R_X(t-u-\gamma) \, d\lambda \, d\gamma = R_Y(t-u)$$

  - Define $R_h(\gamma) = \int h(\lambda) h(\lambda-\gamma) \, d\lambda = h(\lambda) * h(-\lambda)$.
  - Then, $R_Y(\tau) = \int R_h(\gamma) R_X(\tau-\gamma) \, d\gamma = R_h(\tau) * R_X(\tau)$

Exercise: Filtered White Noise Process

- Let the white Gaussian noise process X_t be input to a filter with impulse response

$$h(t) = e^{-at}u(t) = \begin{cases} e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

- Compute the second order description of the output process Y_t.
- **Answers:**
  - Mean: $m_Y = 0$
  - Autocorrelation:

$$R_Y(\tau) = \frac{N_0}{2} \frac{e^{-a|\tau|}}{2a}$$