Power Spectral Density — Concept

- **Power Spectral Density** (PSD) measures how the power of a random process is distributed over frequency.
  - Notation: $S_X(f)$
  - Units: Watts per Hertz (W/Hz)
- Thought experiment:
  - Pass random process X_t through a narrow bandpass filter:
    - center frequency f
    - bandwidth Δf
    - denote filter output as Y_t
  - Measure the power P at the output of bandpass filter:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |Y_t|^2 \, dt$$

  - Relationship between power and (PSD)

$$P \approx S_X(f) \cdot \Delta f.$$

Relation to Autocorrelation Function

- For a wss random process, the power spectral density is closely related to the autocorrelation function $R_X(\tau)$.
- **Definition:** For a random process X_t with autocorrelation function $R_X(\tau)$, the power spectral density $S_X(f)$ is defined as the Fourier transform of the autocorrelation function,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{j2\pi f\tau} \, d\tau.$$

  - For non-stationary processes, it is possible to define a spectral represenattion of the process.
  - However, the spectral contents of a non-stationary process will be time-varying.
- **Example:** If N_t is white noise, i.e., $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$, then

$$S_X(f) = \frac{N_0}{2} \quad \text{for all } f$$

Properties of the PSD

- Inverse Transform:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{-j2\pi f \tau} \, df.$$

- The total power of the process is

$$\mathbf{E}[|X_t|^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) \, df.$$

- $S_X(f)$ is even and non-negative.
  - Evenness of $S_X(f)$ follows from evenness of $R_X(\tau)$.
  - Non-negativeness is a consequence of the autocorrelation function being positive definite

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) f^*(u) R_X(t,u) \, dt \, du \geq 0$$

for all choices of $f(\cdot)$, including $f(t) = e^{-j2\pi ft}$.

Filtering of Random Processes

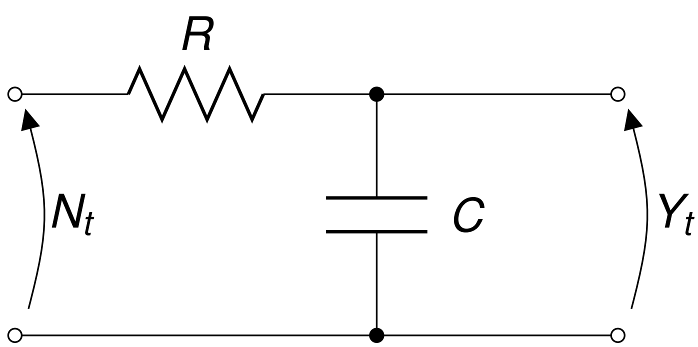
- Random process X_t with autocorrelation $R_X(\tau)$ and PSD $S_X(f)$ is input to LTI filter with impuse response $h(t)$ and frequency response $H(f)$.
- The PSD of the output process Y_t is

$$S_Y(f) = |H(f)|^2 S_X(f).$$

  - Recall that $R_Y(\tau) = R_X(\tau) * C_h(\tau)$,
  - where $C_h(\tau) = h(\tau) * h(-\tau)$.
  - In frequency domain: $S_Y(f) = S_X(f) \cdot \mathcal{F}\{C_h(\tau)\}$
  - With

$$\begin{aligned} \mathcal{F}\{C_h(\tau)\} &= \mathcal{F}\{h(\tau) * h(-\tau)\} \\ &= \mathcal{F}\{h(\tau)\} \cdot \mathcal{F}\{h(-\tau)\} \\ &= H(f) \cdot H^*(f) = |H(f)|^2. \end{aligned}$$

Exercise: Filtered White Noise



- Let N_t be a white noise process that is input to the above circuit. Find the power spectral density of the output process.
- **Answer:**

$$S_Y(f) = \left| \frac{1}{1 + j2\pi fRC} \right|^2 \frac{N_0}{2} = \frac{1}{1 + (2\pi fRC)^2} \frac{N_0}{2}.$$

Signal Space Concepts — Why we Care

- **Signal Space Concepts** are a powerful tool for the analysis of communication systems and for the design of optimum receivers.
- **Key Concepts:**
  - Orthonormal basis functions — tailored to signals of interest — span the signal space.
  - *Representation theorem:* allows any signal to be represented as a (usually finite dimensional) vector
    - Signals are interpreted as points in signal space.
  - For random processes, representation theorem leads to random signals being described by random vectors with uncorrelated components.
    - *Theorem of Irrelavance* allows us to disregrad nearly all components of noise in the receiver.
- We will briefly review key ideas that provide underpinning for signal spaces.

Linear Vector Spaces

- The basic structure needed by our signal spaces is the idea of linear vector space.
- **Definition:** A linear vector space $\mathcal{S}$ is a collection of elements (“vectors”) with the following properties:
  - Addition of vectors is defined and satisfies the following conditions for any $x, y, z \in \mathcal{S}$:
    1. $x + y \in \mathcal{S}$ (closed under addition)
    2. $x + y = y + x$ (commutative)
    3. $(x + y) + z = x + (y + z)$ (associative)
    4. The zero vector $\vec{0}$ exists and $\vec{0} \in \mathcal{S}$. $x + \vec{0} = x$ for all $x \in \mathcal{S}$.
    5. For each $x \in \mathcal{S}$, a unique vector $(-x)$ is also in $\mathcal{S}$ and $x + (-x) = \vec{0}$.

Linear Vector Spaces — continued

- **Definition — continued:**
  - Associated with the set of vectors in $\mathcal{S}$ is a set of scalars. If a, b are scalars, then for any $x, y \in \mathcal{S}$ the following properties hold:
    1. $a \cdot x$ is defined and $a \cdot x \in \mathcal{S}$.
    2. $a \cdot (b \cdot x) = (a \cdot b) \cdot x$
    3. Let 1 and 0 denote the multiplicative and additive identies of the field of scalars, then $1 \cdot x = x$ and $0 \cdot x = \vec{0}$ for all $x \in \mathcal{S}$.
    4. Associative properties:

$$a \cdot (x + y) = a \cdot x + a \cdot y$$
$$(a + b) \cdot x = a \cdot x + b \cdot x$$

Running Examples

- The space of length-N vectors $\mathbb{R}^N$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_N + y_N \end{pmatrix} \text{ and } a \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} a \cdot x_1 \\ \vdots \\ a \cdot x_N \end{pmatrix}$$

- The collection of all square-integrable signals over $[T_a, T_b]$, i.e., all signals $x(t)$ satisfying

$$\int_{T_a}^{T_b} |x(t)|^2 \, dt < \infty.$$

  - Verifying that this is a linear vector space is easy.
  - This space is called $L^2(T_a, T_b)$ (pronounced: ell-two).

Inner Product

- To be truly useful, we need linear vector spaces to provide
  - means to measure the length of vectors and
  - to measure the distance between vectors.
- Both of these can be achieved with the help of inner products.
- **Definition:** The inner product of two vectors $x, y, \in \mathcal{S}$ is denoted by $\langle x, y \rangle$. The inner product is a *scalar* assigned to x and y so that the following conditions are satisfied:
  1. $\langle x, y \rangle = \langle y, x \rangle$ (for complex vectors $\langle x, y \rangle = \langle y, x \rangle^*$)
  2. $\langle a \cdot x, y \rangle = a \cdot \langle x, y \rangle$, with scalar a
  3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, with vector z
  4. $\langle x, x \rangle > 0$, except when $x = \vec{0}$; then, $\langle x, x \rangle = 0$.

Exercise: Valid Inner Products?

- $x, y \in \mathbb{R}^N$ with
$$\langle x, y \rangle = \sum_{n=1}^{N} x_n y_n$$
- **Answer:** Yes; this is the standard *dot product*.
- $x, y \in \mathbb{R}^N$ with
$$\langle x, y \rangle = \sum_{n=1}^{N} x_n \cdot \sum_{n=1}^{N} y_n$$
- **Answer:** No; last condition does not hold, which makes this inner product useless for measuring distances.
- $x(t), y(t) \in L^2(a, b)$ with
$$\langle x(t), y(t) \rangle = \int_a^b x(t) y(t)\, dt$$
- Yes; continuous-time equivalent of the dot-product.

Exercise: Valid Inner Products?

- $x, y \in \mathbb{C}^N$ with

$$\langle x, y \rangle = \sum_{n=1}^{N} x_n y_n^*$$

- **Answer:** Yes; the conjugate complex is critical to meet the last condition (e.g., $\langle j, j \rangle = -1 < 0$).
- $x, y \in \mathbb{R}^N$ with

$$\langle x, y \rangle = x^T K y = \sum_{n=1}^{N} \sum_{m=1}^{N} x_n K_{n,m} y_m$$

with K an $N \times N$-matrix

- **Answer:** Only if K is positive definite (i.e., $x^T K x > 0$ for all $x \neq \vec{0}$).

Norm of a Vector

- **Definition:** The norm of vector $x \in \mathcal{S}$ is denoted by $\|x\|$ and is defined via the inner product as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

  - Notice that $\|x\| > 0$ unless $x = \vec{0}$, then $\|x\| = 0$.
  - The norm of a vector measures the length of a vector.
  - For signals $\|x(t)\|^2$ measures the *energy* of the signal.
- **Example:** For $x \in \mathbb{R}^N$, Cartesian length of a vector

$$\|x\| = \sqrt{\sum_{n=1}^{N} |x_n|^2}$$

Norm of a Vector — continued

- **Illustration:**

$$\|a \cdot x\| = \sqrt{\langle a \cdot x, a \cdot x \rangle} = a\|x\|$$

  - Scaling the vector by a, scales its length by a.

Inner Product Space

- We call a linear vector space with an associated, valid inner product an inner product space.
  - **Definition:** An inner product space is a linear vector space in which a inner product is defined for all elements of the space and the norm is given by $\|x\| = \langle x, x \rangle$.
- **Standard Examples:**
  1. $\mathbb{R}^N$ with $\langle x, y \rangle = \sum_{n=1}^{N} x_n y_n$.
  2. $L^2(a, b)$ with $\langle x(t), y(t) \rangle = \int_a^b x(t) y(t) \, dt$.

Schwartz Inequality

- The following relationship between norms and inner products holds for all inner product spaces.
- **Schwartz Inequality:** For any $x, y \in \mathcal{S}$, where $\mathcal{S}$ is an inner product space,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

with equality if and only if $x = c \cdot y$ with scalar c

  - Proof follows from $\|x + a \cdot y\|^2 \geq 0$ with $a = -\frac{\langle x,y \rangle}{\|y\|^2}$.

Orthogonality

- **Definition:** Two vectors are orthogonal if the inner product of the vectors is zero, i.e.,

$$\langle x, y \rangle = 0.$$

- **Example:** The standard basis vectors e_m in $\mathbb{R}^N$ are orthogonal; recall

$$e_m = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

the 1 occurs on the m-th row

Orthogonality

- **Example:** The basis functions for the Fourier Series expansion $w_m(t) \in L^2(0, T)$ are orthogonal; recall

$$w_m(t) = \frac{1}{\sqrt{T}} e^{j2\pi mt/T}.$$

Distance between Vectors

- **Definition:** The distance d between two vectors is defined as the norm of their difference, i.e.,

$$d(x, y) = \|x - y\|$$

- **Example:** The Cartesian (or Euclidean) distance between vectors in $\mathbb{R}^N$:

$$d(x, y) = \|x - y\| = \sqrt{\sum_{n=1}^{N} |x_n - y_n|^2}.$$

- **Example:** The root-mean-squared error (RMSE) between two signals in $L^2(a, b)$ is

$$d(x(t), y(t)) = \|x(t) - y(t)\| = \sqrt{\int_a^b |x(t) - y(t)|^2 \, dt}$$

Properties of Distances

- Distance measures defined by the norm of the difference between vectors x, y have the following properties:
  1. $d(x, y) = d(y, x)$
  2. $d(x, y) = 0$ if and only if $x = y$
  3. $d(x, y) \leq d(x, z) + d(y, z)$ for all vectors z (Triangle inequality)

Exercise: Prove the Triangle Inequality

- Begin like this:

$$
\begin{aligned}
d^2(x, y) &= \|x - y\|^2 \\
&= \|(x - z) + (z - y)\|^2 \\
&= \langle (x - z) + (z - y), (x - z) + (z - y) \rangle
\end{aligned}
$$

-

$$
\begin{aligned}
d^2(x, y) &= \langle x - z, x - z \rangle + 2\langle x - z, z - y \rangle + \langle z - y, z - y \rangle \\
&\leq \langle x - z, x - z \rangle + 2|\langle x - z, z - y \rangle| + \langle z - y, z - y \rangle \\
&\leq \langle x - z, x - z \rangle + 2\|x - z\| \cdot \|z - y\| + \langle z - y, z - y \rangle \\
&= (d(x, z) + (d(y, z))^2
\end{aligned}
$$

Hilbert Spaces — Why we Care

- We would like our vector spaces to have one more property.
  - We say the sequence of vectors $\{x_n\}$ converges to vector x, if
    $$\lim_{n\to\infty} \|x_n - x\| = 0.$$
  - We would like the limit point x of any sequence $\{x_n\}$ to be in our vector space.
  - Integrals and derivatives are fundamentally limits; we want derivatives and integrals to stay in the vector space.
  - A vector space is said to be closed if it contains all of its limit points.
- **Definition:** A closed, inner product space is A Hilbert Space.

Hilbert Spaces — Examples

- **Examples:** Both $\mathbb{R}^N$ and $L^2(a, b)$ are Hilbert Spaces.
- **Counter Example:** The space of rational number $\mathbb{Q}$ is **not** closed (i.e., not a Hilbert space)

Subspaces

- **Definition:** Let $\mathcal{S}$ be a linear vector space. The space $\mathcal{L}$ is a subspace of $\mathcal{S}$ if
  1. $\mathcal{L}$ is a *subset* of $\mathcal{S}$ and
  2. $\mathcal{L}$ is *closed*.
     - If $x, y \in \mathcal{L}$ then also $x, y, \in \mathcal{S}$.
     - And, $a \cdot x + b \cdot y \in \mathcal{L}$ for all scalars a, b.
- **Example:** Let $\mathcal{S}$ be $L^2(T_a, T_b)$. Define $\mathcal{L}$ as the set of all sinusoids of frequency f_0, i.e., signals of the form $x(t) = A\cos(2\pi f_0 t + \phi)$, with $0 \leq A < \infty$ and $0 \leq \phi < 2\pi$
  1. All such sinusoids are square integrable.
  2. Linear combination of two sinusoids of frequency f_0 is a sinusoid of the same frequency.

Projection Theorem

- **Definition:** Let $\mathcal{L}$ be a subspace of the Hilbert Space $\mathcal{H}$. The vector $x \in \mathcal{H}$ (and $x \notin \mathcal{L}$) is orthogonal to the subspace $\mathcal{L}$ if $\langle x, y \rangle = 0$ for every $y \in \mathcal{L}$.
- **Projection Theorem:** Let $\mathcal{H}$ be a Hilbert Space and $\mathcal{L}$ is a subspace of $\mathcal{H}$.
  Every vector $x \in \mathcal{H}$ has a unique decomposition

$$x = y + z$$

  with $y \in \mathcal{L}$ and z orthogonal to $\mathcal{L}$.
  Furthermore,

$$\|z\| = \|x - y\| = \min_{\nu \in \mathcal{L}} \|x - \nu\|.$$

  - y is called the projection of x onto $\mathcal{L}$.
  - Distance from x to all elements of $\mathcal{L}$ is minimized by y.

Exercise: Fourier Series

- Let $x(t)$ be a signal in the Hilbert space $L^2(0, T)$.
- Define the subspace $\mathcal{L}$ of signals $\nu_n(t) = A_n \cos(2\pi nt/T)$ for a fixed n.
- Find the signal $y(t) \in \mathcal{L}$ that minimizes

$$\min_{y(t)\in\mathcal{L}} \|x(t) - y(t)\|^2.$$

- **Answer:** $y(t)$ is the sinusoid with amplitude

$$A_n = \frac{2}{T}\int_0^T x(t)\cos(2\pi nt/T)\,dt = \frac{2}{T}\langle x(t), \cos(2\pi nt/T)\rangle.$$

  - Note that this is (part of the trigonometric form of) the Fourier Series expansion.
  - Note that the inner product performs the projection of $x(t)$ onto $\mathcal{L}$.