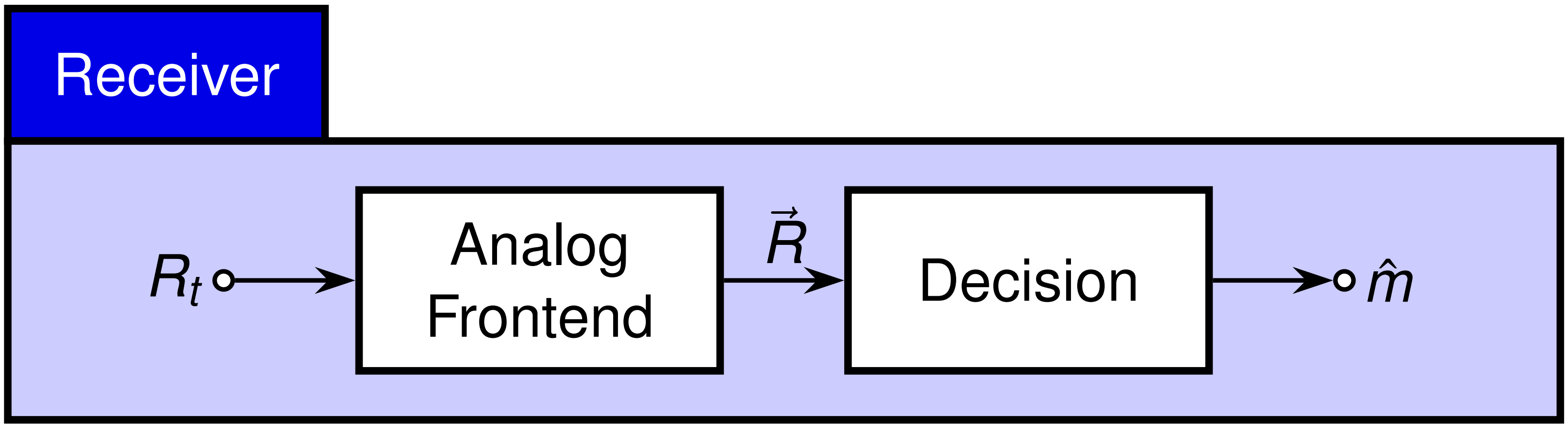
# Structure of a Generic Receiver




- Receivers consist of:
  - an *analog frontend*: maps observed signal $R_t$ to decision statistic $\vec{R}$.
  - *decision device*: determines which symbol $\hat{m}$ was sent based on observation of $\vec{R}$.
- Optimum design of decision device will be considered first.

# Problem Setup

- **Given:**
  - a random vector $\vec{R} \in \mathbb{R}^n$ of observations and
  - hypotheses, $H_0$ and $H_1$, providing statistical models for $\vec{R}$:

$$H_0\colon \vec{R} \sim p_{\vec{R}|H_0}(\vec{r}|H_0)$$

$$H_1\colon \vec{R} \sim p_{\vec{R}|H_1}(\vec{r}|H_1)$$

  with known *a priori* probabilities $\pi_0 = \Pr\{H_0\}$ and $\pi_1 = \Pr\{H_1\}$ ($\pi_0 + \pi_1 = 1$).
- **Problem:** Decide which of the two hypotheses is best supported by the observation $\vec{R}$.
  - Specific objective: minimize the probability of error

$$\begin{aligned}\Pr\{e\} &= \Pr\{\text{decide } H_0 \text{ when } H_1 \text{ is true}\} \\ &\quad + \Pr\{\text{decide } H_1 \text{ when } H_0 \text{ is true}\} \\ &= \Pr\{\text{decide } H_0|H_1\}\Pr\{H_1\} + \Pr\{\text{decide } H_1|H_0\}\Pr\{H_0\}\end{aligned}$$

# Generic Decision Rule

- The decision device performs a mapping that assigns a decision, $H_0$ or $H_1$, to each possible observation $\vec{R} \in \mathbb{R}^n$.
- A generic way to realize such a mapping is:
  - partition the space of all possible observations, $\mathbb{R}^n$, into two disjoint, complementary decision regions $\Gamma_0$ and $\Gamma_1$:

$$\Gamma_0 \cup \Gamma_1 = \mathbb{R}^n \text{ and } \Gamma_0 \cap \Gamma_1 = \emptyset.$$

  - **Decision Rule:**

$$\text{If } \vec{R} \in \Gamma_0\text{: decide } H_0$$

$$\text{If } \vec{R} \in \Gamma_1\text{: decide } H_1$$

# Probability of Error

- The probability of error can now be expressed in terms of the decision regions $\Gamma_0$ and $\Gamma_1$:

$$\begin{aligned}\Pr\{e\} &= \Pr\{\text{decide } H_0|H_1\}\Pr\{H_1\} + \Pr\{\text{decide } H_1|H_0\}\Pr\{H_0\} \\ &= \pi_1 \int_{\Gamma_0} p_{\vec{R}|H_1}(\vec{r}|H_1)\, d\vec{r} + \pi_0 \int_{\Gamma_1} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r}\end{aligned}$$

- Our objective becomes to find the decision regions $\Gamma_0$ and $\Gamma_1$ that minimize the probability of error.

# Probability of Error

- Since $\Gamma_0 \cup \Gamma_1 = \mathbb{R}^n$ it follows that $\Gamma_1 = \mathbb{R}^n \setminus \Gamma_0$

$$\begin{aligned}
\Pr\{e\} &= \pi_1 \int_{\Gamma_0} p_{\vec{R}|H_1}(\vec{r}|H_1)\, d\vec{r} + \pi_0 \int_{\mathbb{R}^n \setminus \Gamma_0} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r} \\
&= \pi_0 \int_{\mathbb{R}^n} p_{\vec{R}|H_0}(\vec{r}|H_0)\, d\vec{r} \\
&+ \int_{\Gamma_0} (\pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1) - \pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0))\, d\vec{r} \\
&= \pi_0 - \int_{\Gamma_0} (\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1))\, d\vec{r}.
\end{aligned}$$

- $\Pr\{e\}$ is minimized by chosing $\Gamma_0$ to contain all $\vec{r}$ for which the integrand $(\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)) < 0$.

## Minimum Pr$\{e\}$ (MPE) Decision Rule

- Thus, the decision region $\Gamma_0$ that minimizes the probability of error is given by:

$$\begin{aligned}
\Gamma_0 &= \left\{\vec{r} : (\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) - \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)) > 0\right\} \\
&= \left\{\vec{r} : \pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0) > \pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1))\right\} \\
&= \left\{\vec{r} : \frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} < \frac{\pi_0}{\pi_1}\right\}
\end{aligned}$$

- The decision region $\Gamma_1$ follows

$$\Gamma_1 = \Gamma_0^C = \left\{\vec{r} : \frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} > \frac{\pi_0}{\pi_1}\right\}$$

# Likelihood Ratio

- The MPE decision rule can be written as

$$\text{If } \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \begin{cases} > \frac{\pi_0}{\pi_1} & \text{decide } H_1 \\ < \frac{\pi_0}{\pi_1} & \text{decide } H_0 \end{cases}$$

  - **Notation:**

$$\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

- The ratio of conditional density functions

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}$$

is called the likelihood ratio.

# Log-Likelihood Ratio

- Many of the densities of interest are exponential functions (e.g., Gaussian).
- For these densities, it is advantageous to take the log of both sides of the decision rule.
  - **Important:** This does not change the decision rule because the logarithm is monotonically increasing!
- The MPE decision rule can be written as:

$$L(\vec{R}) = \ln\left(\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\pi_0}{\pi_1}\right)$$

- $L(\vec{R}) = \ln(\Lambda(\vec{R}))$ is called the log-likelihood ratio.

# Example: Gaussian Hypothesis Testing

- The most important hypothesis testing problem for communications over AWGN channels is

$$H_0 : \vec{R} \sim N(\vec{m}_0, \sigma^2 I)$$
$$H_1 : \vec{R} \sim N(\vec{m}_1, \sigma^2 I)$$

- This problem arises when
  - one of two known signals is transmitted over an AWGN channel, and
  - a linear analog frontend is used.
- Note that
  - the conditional means are different - reflecting different signals
  - covariance matrices are the same - since they depend on noise only.
  - components of $\vec{R}$ are independent - indicating that the frontend projects $R_t$ onto orthogonal bases.

# Resulting Log-Likelihood Ratio

- For this problem, the log-likelihood ratio simplifies to

$$
\begin{aligned}
L(\vec{R}) &= \frac{1}{2\sigma^2} \sum_{k=1}^{n} (R_k - m_{0k})^2 - (R_k - m_{1k})^2 \\
&= \frac{1}{2\sigma^2} (\|\vec{R} - \vec{m}_0\|^2 - \|\vec{R} - \vec{m}_1\|^2) \\
&= \frac{1}{2\sigma^2} \left( 2\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle - (\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2) \right)
\end{aligned}
$$

  - The second expressions shows that the *Euclidean distance* between observations $\vec{R}$ and means $\vec{m}_i$ plays a central role in Gaussian hypothesis testing.
  - The last expression highlights the projection of the observation $\vec{R}$ onto the difference between the means $\vec{m}_i$.

# MPE Decision Rule

- With the above log-liklihood ratio, the MPE decision rule becomes equivalently
  - either

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}$$

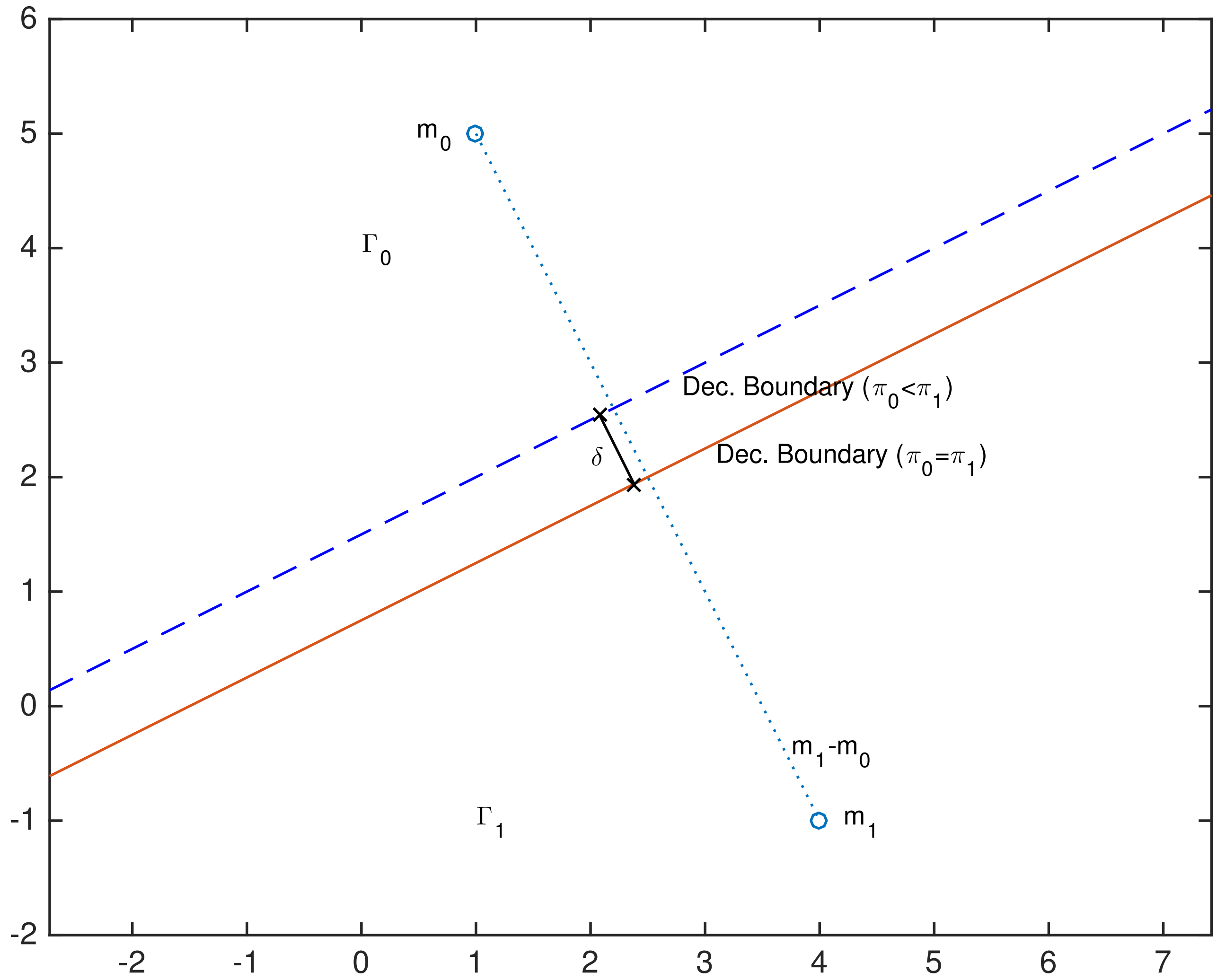
  - or

$$\|\vec{R} - \vec{m}_0\|^2 - 2\sigma^2 \ln(\pi_0) \underset{H_0}{\overset{H_1}{\gtrless}} \|\vec{R} - \vec{m}_1\|^2 - 2\sigma^2 \ln(\pi_1)$$

# Decision Regions

- The MPE decision rule divides $\mathbb{R}^n$ into two half planes that are the decision regions $\Gamma_0$ and $\Gamma_1$.
- The dividing line (decision boundary) between the regions is *perpendicular to* $\vec{m}_1 - \vec{m}_0$.
  - This is a consequence of the inner product in the first form of the decision rule.
- If the priors $\pi_0$ and $\pi_1$ are equal, then the decision boundary passes through the midpoint $\frac{\vec{m}_0+\vec{m}_1}{2}$.
  - For unequal priors, the decision boundary is shifted towards the mean of the *less likely* hypothesis.
  - The distance of this shift equals $\delta = \frac{2\sigma^2|\ln(\pi_0/\pi_1)|}{\|\vec{m}_1-\vec{m}_0\|}$.
  - This follows from the (squared) distances in the second form of the decision rule.

# Decision Regions



# Probability of Error

- **Question:** What is the probability of error with the MPE decision rule?
  - Using MPE decision rule

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}$$

- **Plan:**
  - Find conditional densities of $\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle$ under $H_0$ and $H_1$.
  - Find conditional error probabilities

$$\int_{\Gamma_i} p_{\vec{R}|H_j}(\vec{r}|H_j)\, d\vec{r} \text{ for } i \neq j.$$

  - Find average probability of error.

# Conditional Distributions

- Since $\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle$ is a linear transformation and $\vec{R}$ is Gaussian, the conditional distributions are Gaussian.

$$H_0\colon N(\underbrace{\|\langle \vec{m}_0, \vec{m}_1 \rangle - \vec{m}_0\|^2}_{\mu_0}, \underbrace{\sigma^2 \|\vec{m}_0 - \vec{m}_1\|^2}_{\sigma_m^2})$$

$$H_1\colon N(\underbrace{\|\vec{m}_1\|^2 - \langle \vec{m}_0, \vec{m}_1 \rangle}_{\mu_1}, \underbrace{\sigma^2 \|\vec{m}_0 - \vec{m}_1\|^2}_{\sigma_m^2})$$

# Conditional Error Probabilities

- The MPE decision rule compares

$$\langle \vec{R}, \vec{m}_1 - \vec{m}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\sigma^2 \ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\vec{m}_1\|^2 - \|\vec{m}_0\|^2}{2}}_{\gamma}$$

- Resulting conditional probabilities of error

$$\Pr\{e|H_0\} = Q\left(\frac{\gamma - \mu_0}{\sigma_m}\right) = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} + \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right)$$

$$\Pr\{e|H_1\} = Q\left(\frac{\mu_1 - \gamma}{\sigma_m}\right) = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} - \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right)$$

# Average Probability of Error

- The average error probability equals

$$\begin{aligned}\Pr\{e\} &= \Pr\{\text{decide } H_0|H_1\}\Pr\{H_1\} + \Pr\{\text{decide } H_1|H_0\}\Pr\{H_0\} \\ &= \pi_0 Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} + \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right) + \\ &\quad \pi_1 Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma} - \frac{\sigma \ln(\pi_0/\pi_1)}{\|\vec{m}_0 - \vec{m}_1\|}\right)\end{aligned}$$

- Important special case: $\pi_0 = \pi_1 = \frac{1}{2}$

$$\Pr\{e\} = Q\left(\frac{\|\vec{m}_0 - \vec{m}_1\|}{2\sigma}\right)$$

  - The error probability depends on the ratio of
    - distance between means $\|\vec{m}_0 - \vec{m}_1\|$
    - and noise standard deviation

# Maximum-Likelihood (ML) Decision Rule

- The maximum-likelihood decision rule disregards priors and decides for the hypothesis with higher likelihood.
- **ML Decision rule:**

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

or equivalently, in terms of the log-likelihood,

$$L(\vec{R}) = \ln\left(\frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

- Obviously, the ML decision is equivalent to the MPE rule when the priors are equal.
- In the Gaussian case, the ML rule does not require knowledge of the noise variance.

# A-Posteriori Probability

- By Bayes rule, the probability of hypothesis $H_i$ *after* observing $\vec{R}$ is

$$\Pr\{H_i|\vec{R}=\vec{r}\} = \frac{\pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)}{p_{\vec{R}}(\vec{r})},$$

  where $p_{\vec{R}}(\vec{r})$ is the unconditional pdf of $\vec{R}$

$$p_{\vec{R}}(\vec{r}) = \sum_i \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i).$$

- **Maximum A-Posteriori (MAP) decision rule:**

$$\Pr\{H_1|\vec{R}=\vec{r}\} \underset{H_0}{\overset{H_1}{\gtrless}} \Pr\{H_0|\vec{R}=\vec{r}\}$$

  - **Interpretation:** Decide in favor of the hypothesis that is more likely given the observed signal $\vec{R}$.

# The MAP and MPE Rules are Equivalent

- The MAP and MPE rules are equivalent: the MAP decision rule achieves the minimum probability of error.
- The MAP rule can be written as

$$\frac{\Pr\{H_1|\vec{R}=\vec{r}\}}{\Pr\{H_0|\vec{R}=\vec{r}\}} \underset{H_0}{\overset{H_1}{\gtrless}} 1.$$

- Inserting $\Pr\{H_i|\vec{R}=\vec{r}\} = \frac{\pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)}{p_{\vec{R}}(\vec{r})}$ yields

$$\frac{\pi_1 p_{\vec{R}|H_1}(\vec{r}|H_1)}{\pi_0 p_{\vec{R}|H_0}(\vec{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

  - This is obviously equal to the MPE rule

$$\frac{p_{\vec{R}|H_1}(\vec{r}|H_1)}{p_{\vec{R}|H_0}(\vec{r}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}.$$

# More than Two Hypotheses

- Frequently, more than two hypotheses must be considered:

$$
\begin{aligned}
H_0 &: \vec{R} \sim p_{\vec{R}|H_0}(\vec{r}|H_0) \\
H_1 &: \vec{R} \sim p_{\vec{R}|H_1}(\vec{r}|H_1) \\
&\vdots \\
H_M &: \vec{R} \sim p_{\vec{R}|H_M}(\vec{r}|H_M)
\end{aligned}
$$

- In these cases, it is no longer possible to reduce the decision rules to
  - the computation of the likelihood ratio
  - followed by comparison to a threshold

# More than Two Hypotheses

- Instead the decision rules take the following forms
  - **MPE rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i)$$

  - **ML rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} p_{\vec{R}|H_i}(\vec{r}|H_i)$$

  - **MAP rule:**

$$\hat{m} = \arg \max_{i \in \{0,\ldots,M-1\}} \Pr\{H_i|\vec{R} = \vec{r}\}$$

# More than Two Hypotheses: The Gaussian Case

- When the hypotheses are of the form $H_i$: $\vec{R} \sim N(\vec{m}_i, \sigma^2 I)$, then the decision rules become:
  - **MPE and MAP decision rules:**

$$\hat{m} = \arg \min_{i \in \{0,\ldots,M-1\}} \|\vec{r} - \vec{m}_i\|^2 - 2\sigma^2 \ln(\pi_i)$$

$$= \arg \max_{i \in \{0,\ldots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle + \sigma^2 \ln(\pi_i) - \frac{\|\vec{m}_i\|^2}{2}$$

  - **ML decision rule:**

$$\hat{m} = \arg \min_{i \in \{0,\ldots,M-1\}} \|\vec{r} - \vec{m}_i\|^2$$

$$= \arg \max_{i \in \{0,\ldots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle - \frac{\|\vec{m}_i\|^2}{2}$$

    - This is also the MPE rule when the priors are all equal.

# Take-Aways

- The conditional densities $p_{\vec{R}|H_i}(\vec{r}|H_i)$ play a key role.
- **MPE decision rule:**
  - Binary hypotheses:

$$\Lambda(\vec{R}) = \frac{p_{\vec{R}|H_1}(\vec{R}|H_1)}{p_{\vec{R}|H_0}(\vec{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

  - $M$ hypotheses:

$$\hat{m} = \arg \max_{i \in \{0,\dots,M-1\}} \pi_i p_{\vec{R}|H_i}(\vec{r}|H_i).$$

# Take-Aways

- For the Gaussian case (different means, equal variance), decisions are based on the Euclidean distance between observations $\vec{R}$ and conditional means $\vec{m}_i$:

$$
\begin{aligned}
\hat{m} &= \arg \min_{i \in \{0,\dots,M-1\}} \|\vec{r} - \vec{m}_i\|^2 - 2\sigma^2 \ln(\pi_i) \\
&= \arg \max_{i \in \{0,\dots,M-1\}} \langle \vec{r}, \vec{m}_i \rangle + \sigma^2 \ln(\pi_i) - \frac{\|\vec{m}_i\|^2}{2}
\end{aligned}
$$